$f(0) = 0 \Rightarrow u(0,0) + i v(0,0) = 0$

$\therefore u(0,0) = 0, \quad v(0,0) = 0$

$$\left(\frac{\partial u}{\partial x}\right)_{z=0} = \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h}$$

$$= \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{z=0} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{-y-0}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{z=0} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x-0}{x} = 1$$

$$\left(\frac{\partial v}{\partial y}\right)_{z=0} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{y-0}{y} = 1$$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at $z=0$ proved

Third part:- we have now to show that $f'(0)$ does not exist

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z\to 0} \frac{f(z) - 0}{z}$$

③ ②

Third part: ~~we have to show that~~

17/1/2021

~~$f'(0)$~~

$$= \lim_{z\to 0} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

Let $z \to 0$ along the path $y=x$

Then $f'(0) = \lim_{x\to 0} \frac{x^3-x^3+i(x^3+x^3)}{(x^2+x^2)(x+ix)}$

$$= \frac{i\,2x^3}{2x^2 \cdot x(1+i)} = \frac{i}{1+i} \quad (1)$$

Let $z \to 0$ along x-axis [here $y=0$]

$\therefore f'(x) = \lim_{x\to 0} \frac{x^3-0+i(x^3+0)}{(x^2+0)(x+i\cdot 0)}$ [$\because y=0$]

$$= \frac{(1+i)x^3}{x^3} = (1+i)$$

Thus $f'(0) = \frac{\cancel{2}i}{\cancel{2}(1+i)}$ along the path $y=x$

$= (1+i)$ along the path $y=0$

Since the values of $f'(0)$ are not unique along different paths, hence $f'(0)$ does not exist. So $f(z)$ is not analytic at $z=0$ proved

B.Sc part I Math(H)
paper - I, Group-A
Hyperbolic function
Dr. Gulshan Dayal singh
17.04.2021

(6) If $\cos(x+iy) = \cos\theta + i\sin\theta$
prove $\cos 2x + \cosh 2y = 2$

$\because \cos x \cdot \cos iy - \sin x \cdot \sin iy = \cos\theta + i\sin\theta$

$\therefore \cos x \cdot \cosh y - i\sin x \sinh y = \cos\theta + i\sin\theta$.

Equating real and imaginary parts,

$\cos x \cosh y = \cos\theta$ — (1)

$-\sin x \sinh y = \sin\theta$ — (2)

Squaring and adding,

$\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = \cos^2\theta + \sin^2\theta$

or $\frac{1}{2\times 2}\{2\cos^2 x \cdot 2\cosh^2 y + 2\sin^2 x \cdot 2\sinh^2 y\} = 1$

or $\frac{1}{4}\{(1+\cos 2x)(1+\cosh 2y) + (1-\cos 2x)(\cosh 2y - 1)\} = 1$

or $\{1 + \cosh 2y + \cos 2x + \cos 2x \cosh 2y + \cosh 2y - 1 - \cos 2x \cosh 2y + \cos 2x\} = 4$

or $2\cosh 2y + 2\cos 2x = 4$

or $\cosh 2y + \cos 2x = 2$ proved

(7.) If $\sin(A+iB) = x+iy$

prove $\frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1$.

From question,

$\sin A \cdot \cos iB + \cos A \sin iB = x+iy$

or $\sin A \cosh B + i\cos A \sinh B = x+iy$

Equating real and imaginary part, we have

$x = \sin A \cdot \cosh B$ (1)

$y = \cos A \sinh B$ (2)

$x^2 = \sin^2 A \cosh^2 B$

$y^2 = \cos^2 A \sinh^2 B$

$\therefore \dfrac{x^2}{\sin^2 A} = \cosh^2 B$ — (3)

$\dfrac{y^2}{\cos^2 A} = \sinh^2 B$ — (4)

(3) − (4) $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B = 1$ proved

(8) If $\tan(u+iv) = x+iy$

prove that $x^2+y^2+1 = 2y \coth 2v$

$\because \tan(u+iv) = x+iy$

$\therefore \tan(u-iv) = x-iy$

$\therefore u+iv = \tan^{-1}(x+iy)$

$u-iv = \tan^{-1}(x-iy)$

$2iv = \tan^{-1}(x+iy) - \tan^{-1}(x-iy)$

or $2iv = \tan^{-1} \dfrac{x+iy-(x-iy)}{1+(x+iy)(x-iy)}$

or $2iv = \tan^{-1} \dfrac{x+iy-(x-iy)}{1+(x+iy)(x-iy)}$

or $2iv = \tan^{-1} \dfrac{x+iy-x+iy}{1+(x^2+y^2)}$

$\therefore \tan 2iv = \dfrac{2iy}{1+x^2+y^2}$

or $i \tanh 2v = \dfrac{i2y}{1+x^2+y^2}$

or $\tanh 2v = \dfrac{2y}{1+x^2+y^2}$

or $\coth 2v = \dfrac{x^2+y^2+1}{2y}$

or $2y \coth 2v = x^2+y^2+1$ proved